Local and non-local conserved currents for an equation related to the nonlinear σ-model

1982 J. Phys. A: Math. Gen. 15 2349

(http://iopscience.iop.org/0305-4470/15/8/014)

J. Phys. A: Math. Gen. 15 (1982) 2349–2354. Printed in Great Britain

Local and non-local conserved currents for an equation related to the nonlinear σ-model

F J Chinea and F Guil

Departamento de Métodos Matemáticos de la Física, Facultad de Ciencias Físicas, Universidad Complutense de Madrid, Madrid-3, Spain

Received 27 October 1981, in final form 29 March 1982

Abstract. The field equation for a classical nonlinear O(2, 1)-invariant σ-model in two dimensions may be written as the single scalar equation $z_{uv} = 2z_u z_v(z+\bar{z})^{-1}$ for a complex field z. Two infinite families of conserved currents for this equation are obtained by means of its representation as the integrability condition of a linear system of equations. One of the resulting families is of non-local type.

1. Introduction

The equation

$$z_{uv} = 2z_u z_v/(z+\bar{z})$$

is closely related to the classical O(2, 1)-invariant nonlinear σ-model in two dimensions. It may be represented as the integrability condition for a first-order linear system of partial differential equations or, equivalently, by means of a connection with vanishing curvature on an appropriate principal bundle with two-dimensional base space. This equation possesses Bäcklund transformations (Chinea 1981a,b) and an infinite number of conserved currents; it is the purpose of the present note to show the last property. The construction of the conserved currents is based on the representation (2.2), as well as on the gauge covariance of such a system. This allows us to obtain an equivalent linear system (2.6), which may be formally solved by means of the Riccati equation (3.2). The coefficients of appropriate power expansions in a real parameter k of the solution to the Riccati equation give rise to the conserved currents (for other applications of this method see Guil (1982)). It is of interest to notice that two types of power expansions may be introduced, one in negative powers and the other in positive powers of k. The expansion in negative powers yields currents of the usual type, depending on the field and its derivatives. The expansion in positive powers, however, leads to conservation laws containing non-local quantities. This type of non-local current has been recently considered in connection with equations related to (2.1) (see, for instance, Sasaki and Bullough 1981, Eichenherr 1981).

2. Integrability conditions and gauge invariance

The representation of the equation

$$z_{uv} = 2z_u z_v/(z+\bar{z}) \tag{2.1}$$

as the integrability condition for a linear system of equations has been given in terms of the local components of a connection (with zero curvature) on an SU(2) bundle (Chinea 1981 a,b); a similar formulation exists using an SU(1, 1) connection. In (2.1) and in the sequel, a bar denotes complex conjugation, and subscripts correspond to partial derivatives; z is a complex function of u and v.

For the purposes of the present work, the linear system may be written as

$$\varphi_u = M\varphi, \qquad \varphi_v = N\varphi \tag{2.2}$$

where

$$M = am(k) - \bar{a}m^{\mathrm{T}}(k), \qquad N = bm(-k^{-1}) - \bar{b}m^{\mathrm{T}}(-k^{-1}), \qquad \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

with

$$a = (z+\bar{z})^{-1}z_u, \qquad b = (z+\bar{z})^{-1}z_v,$$

$$m(k) = \begin{pmatrix}\frac{1}{2} & k\\ 0 & -\frac{1}{2}\end{pmatrix}, \qquad k \text{ a real constant}$$

(the transpose of a matrix m is denoted by m^{T}).

The integrability condition for system (2.2) may be written as

$$[\partial_u - M, \partial_v - N] = 0 \tag{2.3}$$

which is equivalent to equation (2.1) by construction.

Equation (2.1) may be related to the field equation for an O(2, 1)-invariant nonlinear σ-model (see Pohlmeyer (1976) for a description of the O(3)-invariant case; see also Eichenherr (1981) and references quoted therein for a geometric analysis of nonlinear σ-models defined on symmetric spaces). The field equation is

$$q_{uv} - (q_u \cdot q_v)q = 0 \tag{2.4}$$

where q is a three-dimensional vector, and the scalar product is defined by a diag$(++-)$ metric. The field q is required to satisfy the constraint

$$q^2 = -1. \tag{2.5}$$

Equation (2.1) is shown to be equivalent to (2.4) with (2.5) by introducing the parametrisation

$$q = (z+\bar{z})^{-1}(\mathrm{i}(z-\bar{z}), 1 - z\bar{z}, 1 + z\bar{z}).$$

We refer the reader to Lüscher and Pohlmeyer (1978), Pohlmeyer (1980) and Kafiev (1981) for a detailed account of the properties of nonlinear σ-models and their relation with four-dimensional Yang–Mills fields.

Equation (2.3) has the property of being invariant under appropriate gauge transformations. This is a consequence of the fact that (2.3) represents a (gauge-invariant) zero curvature condition for a connection defined by M and N. We shall use the possibility to modify the form of the matrices M and N in order to obtain a representation equivalent to (2.2), which will be more convenient in what follows.

If a new column vector ψ, related to the previous φ, is defined by

$$\varphi = G\psi$$

where G is a non-singular square matrix, then ψ will satisfy the equations

$$\psi_u = \tilde{M}\psi, \qquad \psi_v = \tilde{N}\psi \tag{2.6}$$

where

$$\tilde{M} = G^{-1}MG - G^{-1}G_u, \qquad \tilde{N} = G^{-1}NG - G^{-1}G_v. \tag{2.7}$$

The freedom allowed by the arbitrariness of G will be used in such a way that equations (2.6) be separable, in the sense that each one of the components of ψ independently satisfies a second-order equation. By differentiating the first equation in (2.6) with respect to u, one gets

$$\psi_{uu} = Q\psi, \qquad Q = \tilde{M}_u + \tilde{M}^2.$$

Writing down Q in terms of G and M,

$$Q = -G^{-1}G_{uu} + 2(G^{-1}G_u)^2 + G^{-1}M^2G + G^{-1}(M_u - 2G_uG^{-1}M)G. \tag{2.8}$$

In order to decouple the equations for ψ_1 and ψ_2, Q must be diagonal. It is easy to see that to ensure the last requirement, it is sufficient to take

$$G = \begin{pmatrix} a^{1/2} & 0 \\ 0 & \bar{a}^{1/2} \end{pmatrix} \tag{2.9}$$

as this guarantees that $M_u - 2G_uG^{-1}M$ in (2.8) is diagonal; the remaining terms are trivially seen to be diagonal (in particular, M^2 is diagonal by virtue of M having vanishing trace). With the choice (2.9) for G, the resulting expression for Q is

$$Q = \begin{pmatrix} l - k^2|a|^2 & 0 \\ 0 & \bar{l} - k^2|a|^2 \end{pmatrix}$$

where $l = \frac{1}{2}[(a - \bar{a}) - a^{-1}a_u]_u + \frac{1}{4}[(a - \bar{a}) - a^{-1}a_u]^2$ and the second-order equations for the components of ψ are

$$\psi_{1uu} = (l - k^2|a|^2)\psi_1, \qquad \psi_{2uu} = (\bar{l} - k^2|a|^2)\psi_2. \tag{2.10}$$

The explicit form of the matrices M and N is given here for the sake of completeness:

$$\tilde{M} = \begin{pmatrix} \frac{1}{2}(a - \bar{a}) - \frac{1}{2}a^{-1}a_u & k|a| \\ -k|a| & -\frac{1}{2}(a - \bar{a}) - \frac{1}{2}\bar{a}^{-1}\bar{a}_u \end{pmatrix}, \qquad \tilde{N} = \begin{pmatrix} 0 & -k^{-1}|a|a^{-1}b \\ k^{-1}|a|\bar{a}^{-1}\bar{b} & 0 \end{pmatrix}.$$

3. Conserved currents

The representation of equation (2.1) by means of the linear system (2.6) will permit the construction of two infinite families of conserved currents for (2.1). Let us consider the function ρ defined by

$$\rho = \psi_1^{-1}\psi_{1u}. \tag{3.1}$$

According to the first equation in (2.6), ρ will satisfy the Riccati equation

$$\rho_u + \rho^2 = l - k^2|a|^2. \tag{3.2}$$

Conversely, a solution ρ of (3.1) defines ψ_1 and ψ_2 as

$$\psi_1 = \exp(\partial_u^{-1}\rho), \qquad \psi_2 = \tfrac{1}{2}k^{-1}|a|^{-1}[2\rho - (a - \bar{a}) + a^{-1}a_u]\exp(\partial_u^{-1}\rho), \tag{3.3}$$

where the first equation in (2.6) has been used. Upon substitution of (3.3) into the second equation, the following continuity equation is obtained:

$$\partial_v\rho+\partial_u\{\tfrac{1}{2}k^{-2}ba^{-1}[2\rho-(a-\bar{a})+a^{-1}a_u]\}=0. \tag{3.4}$$

Notice that choosing ψ_1 or ψ_2 in the definition (3.1) of ρ is irrelevant. It is also clear that everything that is said in terms of the u derivative may be translated into similar equations using the v derivative and *vice versa*, due to the symmetry of the starting equation (2.1) in u and v.

Equation (3.2) admits two types of solutions in terms of formal power series in the parameter k. The first one is of the form

$$\rho=\varepsilon k|a|+\sum_{r\geqslant 0}\chi_r k^{-r} \tag{3.5}$$

with $\varepsilon^2=-1$. The coefficients χ_r are found recursively after substituting ρ as given by (3.5) in equation (3.2), thus obtaining

$$\chi_0=-\tfrac{1}{2}|a|^{-1}|a|_u, \qquad \chi_1=-\tfrac{1}{2}\varepsilon|a|^{-1}(l-\chi_{0u}-\chi_0^2),$$

and, in general

$$\chi_{r+2}=\tfrac{1}{2}\varepsilon|a|^{-1}\left(\chi_{r+1,u}+\sum_{s=0}^{r+1}\chi_{r+1-s}\chi_s\right), \qquad r=0,1,\ldots.$$

Each coefficient of the expansion (3.5) for ρ gives rise to a conserved current of equation (2.1), when equation (3.4) is used. The following infinite set of relations is obtained:

$$\begin{aligned}&\partial_v|a|=0, \qquad \partial_v\chi_0=0, \qquad \partial_v\chi_1+\partial_u(\varepsilon ba^{-1}|a|)=0,\\&\partial_v\chi_2+\partial_u\{\tfrac{1}{2}ba^{-1}[2\chi_0-(a-\bar{a})+a^{-1}a_u]\}=0,\\&\partial_v\chi_{r+3}+\partial_u(ba^{-1}\chi_{r+1})=0, \qquad r=0,1,\ldots.\end{aligned} \tag{3.6}$$

Notice that the first equation in (3.6) shows that $|a|$ is a function of u only. Similarly, $|b|$ will be a function of v only. Recalling the definition of a and b and using the conformal invariance of (2.1) under $u\mapsto U(u)$, $v\mapsto V(v)$, one may set

$$|a|=1, \qquad |b|=1$$

(excluding degenerate cases). We may thus define

$$a=\mathrm{e}^{\mathrm{i}\alpha(u,v)}, \qquad b=\mathrm{e}^{\mathrm{i}\beta(u,v)}.$$

When the definitions of a and b are taken into account, a consistency requirement forces the real functions α and β to satisfy

$$\beta_u=2\sin\alpha, \qquad \alpha_v=2\sin\beta$$

which imply that $\alpha+\beta$ and $\alpha-\beta$ are a pair of conjugate solutions of the sine-Gordon equation

$$\varphi_{uv}=4\sin\varphi.$$

With the chosen normalisation for $|a|$ and $|b|$, $G\in\mathrm{SU}(2)$ and the transformation (2.7) is an SU(2) transformation of the connection (M,N).

Let us now introduce an alternative expansion of

$$\rho=\sum_{r\geqslant 0}\sigma_r k^r. \tag{3.7}$$

As in the previous case, each coefficient σ_r in the expansion (3.7) will give rise to a conserved current of (2.1). Substituting (3.7) into (3.2), the following result is obtained:

$$\sigma_0 = \tfrac{1}{2}(a - \bar{a}) - \tfrac{1}{2}a^{-1}a_u, \qquad \sigma_1 = \exp(-2\partial_u^{-1}\sigma_0),$$
$$\sigma_2 = -\sigma_1\partial_u^{-1}[\sigma_1^{-1}(|a|^2 + \sigma_1^2)], \tag{3.8}$$
$$\sigma_{r+3} = -\sigma_1\partial_u^{-1}\left(\sigma_1^{-1}\sum_{s=1}^{r+2}\sigma_{r+3-s}\sigma_s\right), \qquad r = 0, 1, \ldots .$$

Equation (3.4) may now be used to get the infinite family of conservation laws

$$\partial_v\sigma_r + \partial_u(ba^{-1}\sigma_{r+2}) = 0. \tag{3.9}$$

The presence of the operator ∂_u^{-1} in the σ's appearing in (3.9) makes it clear that the family under consideration is a non-local one.

Finally, let us point out that the term 'conserved current' is not completely appropriate when applied to equation (3.9), which should rather be considered as an infinite set of evolution equations for certain non-local quantities. The reason for this is that the term in parentheses in (3.9) may be shown to be equal to ∂_u^{-1} of an appropriate function. This may easily be concluded by noticing that

$$\sigma_0 = \tfrac{1}{2}\partial_u \ln(ba^{-1})$$

which is a consequence of

$$a - \bar{a} = \partial_u \ln b$$

where (2.1) has been taken into account. This implies that

$$ab^{-1} = \sigma_1$$

and, as a consequence,

$$ba^{-1}\sigma_2 = \sigma_1^{-1}\sigma_2 = -\partial_u^{-1}[\sigma_1^{-1}(|a|^2 + \sigma_1^2)]$$

and, for $r \geq 0$,

$$ba^{-1}\sigma_{r+3} = \sigma_1^{-1}\sigma_{r+3} = -\partial_u^{-1}\left(\sigma_1^{-1}\sum_{s=1}^{r+2}\sigma_{r+3-s}\sigma_s\right).$$

A similar cancellation of the ∂_u and ∂_u^{-1} operators may be shown to appear in the sets of non-local conservation laws quoted above (Sasaki and Bullough 1981, Eichenherr 1981).

Acknowledgment

The authors wish to thank Professor A Galindo for conversations.

References

Chinea F J 1981a *Phys. Rev.* D **24** 1053
—— 1981b *Lett. Math. Phys.* **5** 419
Eichenherr H 1981 *CERN Preprint TH-3073*
Guil F 1982 *J. Math. Phys.* **23** 211

Kafiev Yu N 1981 *Nucl. Phys.* B **178** 177
Lüscher M and Pohlmeyer K 1978 *Nucl. Phys.* B **137** 46
Pohlmeyer K 1976 *Commun. Math. Phys.* **46** 207
—— 1980 *Commun. Math. Phys.* **72** 37
Sasaki R and Bullough R K 1981 *Proc. R. Soc.* A **376** 401